BRIEF COMMUNICATIONS

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Description of the pressure effects in the Reynolds stress transport equations

Johan Groth
Department of Mechanics, Royal Institute of Technology, S-100 44 Stockholm, Sweden

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Three different proposals for the separation into a deviatoric and a nondeviatoric term of the pressure–velocity interactions in the Reynolds stress transport equations are discussed. It is shown from general considerations and two specific examples that the customary separation is likely to be the correct choice.

Since many turbulence models take their starting point in the transport equations for the Reynolds stresses $\overline{u_i u_j}$, it is of great importance that these equations are written in an unambiguous way, and that the different terms in the equations are given their correct physical interpretations. In an incompressible turbulent flow, the Reynolds stress transport (RST) equations are usually derived by multiplication of the transport equation for the fluctuating velocity u_i with u_j, interchanging indices, adding the two expressions, and taking the average. The fluctuating pressure p thus appears in the resulting RST equations as

$$-\overline{u_i \frac{1}{\rho}\frac{\partial p}{\partial x_j}} - \overline{u_j \frac{1}{\rho}\frac{\partial p}{\partial x_i}}. \tag{1}$$

This term is generally separated into a deviatoric term (D_{ij}) and a nondeviatoric or, since it is commonly written as a divergence, transport term (T_{ij}). Having a zero trace D_{ij} does not enter into the transport equation for the turbulent kinetic energy (k), and might be interpreted as an intercomponent energy transfer. The trace of the nondeviatoric term contributes to the change of k and can be interpreted as the work done by the pressure field.

Keeping the continuity equation in mind, the "classical" separation (see, e.g., Ref. 1) is written as

$$\begin{aligned} D_{ij}^{C} &= \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}, \\ T_{ij}^{C} &= \frac{\partial}{\partial x_k}\left(-\overline{\frac{p}{\rho}(u_i\delta_{kj} + u_j\delta_{ki})}\right). \end{aligned} \tag{2}$$

Lumley[2] showed that this separation is not (in a mathematical sense) unique and suggested the separation

$$D_{ij}^{L} = -\left[\overline{\frac{1}{\rho}\left(u_i\frac{\partial p}{\partial x_j} + u_j\frac{\partial p}{\partial x_i}\right)} - \frac{2}{3\rho}\frac{\partial}{\partial x_n}\overline{pu_n}\,\delta_{ij}\right] \tag{3}$$

and

$$T_{ij}^{L} = -\frac{2}{3\rho}\frac{\partial}{\partial x_n}\overline{pu_n}\,\delta_{ij}.$$

However, in a subsequent paper, while constructing an equation for the joint characteristic functional for the fluctuating velocity and temperature fields, Lumley[3] finds reason to return to the "classical" separation. A third separation has been proposed by Mansour *et al.*[4]

$$D_{ij}^{M} = -\left[\overline{\frac{1}{\rho}\left(u_i\frac{\partial p}{\partial x_j} + u_j\frac{\partial p}{\partial x_i}\right)} - \frac{\overline{u_i u_j}}{k}\frac{1}{\rho}\frac{\partial}{\partial x_n}\overline{pu_n}\right] \tag{4}$$

and

$$T_{ij}^{M} = -\frac{\overline{u_i u_j}}{k}\frac{1}{\rho}\frac{\partial}{\partial x_n}\overline{pu_n}.$$

The question that arises is the following: Which, if any, of these three separations is correct in the sense that D_{ij} and T_{ij} have clear physical interpretations?

It is our belief that the confusion, as expressed by (2)–(4), is due to the method used (and outlined above) in obtaining the RST equations. In the process, the origins of the different terms get lost, which leads to a mix of physically different terms [like (1)]. This can be avoided by carefully studying the relevant equations and postponing the introduction of a specific form for the constitutive relation.

We begin here by considering Newton's second law for an infinitesimal cube of fluid, with uniform and constant density, which is unaffected by external body forces

$$\rho\left(\frac{\partial \mathscr{U}_i}{\partial t} + \mathscr{U}_n\frac{\partial \mathscr{U}_i}{\partial x_n}\right) = \frac{\partial}{\partial x_n}\mathscr{S}_{ni}. \tag{5}$$

Here, $\mathscr{U}_i = U_i + u_i$ is the total velocity vector and $\mathscr{S}_{ni} = \Sigma_{ni} + \sigma_{ni}$ is the stress in the i direction acting on a surface element perpendicular to a normal vector in the n direction. Subtracting the equation for the mean velocity

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[the average of (5)] from this expression, we obtain the equation for the fluctuating velocity

$$\rho\left(\frac{\partial u_i}{\partial t}+U_n\frac{\partial u_i}{\partial x_n}+u_n\frac{\partial U_i}{\partial x_n}+\frac{\partial}{\partial x_n}\left(u_i u_n-\overline{u_i u_n}\right)\right)=\frac{\partial}{\partial x_n}\sigma_{ni}. \tag{6}$$

We next consider that part of the work (per unit volume and time), done on the surface of the fluid element by the fluctuating quantities, that is due to movements in the α direction. This work is the difference between the total work in the α direction (see, e.g., Ref. 5, p. 62) and the work done by the mean fields. We obtain (no summation over Greek indices)

$$\begin{aligned} w&=\frac{\partial}{\partial x_n}\left(\mathscr{U}_\alpha\mathscr{S}_{n\alpha}\right)-\frac{\partial}{\partial x_n}\left(U_\alpha\Sigma_{n\alpha}\right)\\ &=\frac{\partial}{\partial x_n}\left(u_\alpha\Sigma_{n\alpha}+U_\alpha\sigma_{n\alpha}+u_\alpha\sigma_{n\alpha}\right). \end{aligned}\tag{7}$$

The mean value of the work done by the fluctuating quantities can thus be written

$$\overline{w}=\frac{\partial}{\partial x_n}\overline{u_\alpha\sigma_{n\alpha}}=\overline{u_\alpha\frac{\partial\sigma_{n\alpha}}{\partial x_n}}+\overline{\sigma_{n\alpha}\frac{\partial u_\alpha}{\partial x_n}}. \tag{8}$$

Using (6) in the second to last term of (8), we obtain

$$\begin{aligned} \overline{w}=&\rho\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_\alpha u_\alpha}\right)+\rho U_n\frac{\partial}{\partial x_n}\left(\frac{1}{2}\overline{u_\alpha u_\alpha}\right)+\rho\,\overline{u_\alpha u_n}\frac{\partial U_\alpha}{\partial x_n}\\ &+\rho\frac{\partial}{\partial x_n}\left(\frac{1}{2}\overline{u_\alpha u_\alpha u_n}\right)+\overline{\sigma_{n\alpha}\frac{\partial u_\alpha}{\partial x_n}}, \end{aligned}\tag{9}$$

i.e., the mean value of the work done by the fluctuating quantities on the surface of the fluid element is balanced by a rate of change of the element's kinetic energy in the α direction, an advection of kinetic energy (by the mean field), an exchange of energy with the mean velocity field, a transport of energy by the velocity fluctuations, and, finally, by a term that must contain all other energy changing processes. This last term includes friction losses (dissipative terms) and effects of an intercomponent energy transfer.

If we now introduce the constitutive relation for a Newtonian fluid, we obtain

$$\begin{aligned} \Sigma_{\text{in}}&=-P\delta_{\text{in}}+\mu\left(\frac{\partial U_i}{\partial x_n}+\frac{\partial U_n}{\partial x_i}\right),\\ \sigma_{\text{in}}&=-p\delta_{\text{in}}+\mu\left(\frac{\partial u_i}{\partial x_n}+\frac{\partial u_n}{\partial x_i}\right). \end{aligned}\tag{10}$$

We may then express the work term as

$$\overline{w}=\frac{\partial}{\partial x_n}\overline{u_\alpha\sigma_{n\alpha}}=\frac{\partial}{\partial x_n}\left[-\overline{pu_\alpha}\delta_{\alpha n}+\mu\,\overline{u_\alpha\left(\frac{\partial u_\alpha}{\partial x_n}+\frac{\partial u_n}{\partial x_\alpha}\right)}\right] \tag{11}$$

and the dissipative/energy transfer term as

$$\overline{\sigma_{n\alpha}\frac{\partial u_\alpha}{\partial x_n}}=-\overline{p\frac{\partial u_\alpha}{\partial x_\alpha}}+\mu\,\overline{\frac{\partial u_\alpha}{\partial x_n}\left(\frac{\partial u_\alpha}{\partial x_n}+\frac{\partial u_n}{\partial x_\alpha}\right)}. \tag{12}$$

Hence direct derivation of the work done by the fluctuating quantities on a fluid element yields a natural split of the pressure related terms into a traceless part, $\overline{p(\partial u_\alpha/\partial x_\alpha)}$ (sum over α), and a divergence term $-(\partial/\partial x_n)(\overline{pu_\alpha}\,\delta_{\alpha n})$. This strongly supports the classical interpretation of the pressure-strain rate correlation term D^C_{ij} as the intercomponent energy transfer term and of T^C_{ij} as the pressure-dependent transport term. Thus, of the three different separations, Eqs. (2)–(4), D^C_{ij} and T^C_{ij} seems to be the physically appropriate choice.

There are also other indications that expression (2) is the correct choice. In two-dimensional turbulence where, say U_3 and u_3 are identically zero, we get, in the corresponding RST equation ($i=j=3$), that both D^C_{33} and T^C_{33} are identically zero. However, from (3) we obtain

$$D^{\text{L}}_{33}=\frac{2}{3\rho}\frac{\partial}{\partial x_n}\overline{pu_n}\quad\text{and}\quad T^{\text{L}}_{33}=-\frac{2}{3\rho}\frac{\partial}{\partial x_n}\overline{pu_n}. \tag{13}$$

That is to say, in the expression proposed in Ref. 2 there is a net work done by the pressure that generates energy in the 3 component. This energy is immediately (as it must be since u_3 is identically zero) transferred to the other components. This later scenario seems rather unphysical. Also, in a flow that is homogeneous in one direction, such as a turbulent boundary layer with spanwise homogeneity, T^C_{33} vanishes identically. The other two expressions, T^{L}_{33} and T^{M}_{33}, indicate a nonzero (i.e., unphysical) transport in the homogeneous direction. Once again, there is support for the classical separation of the pressure related terms.

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