An algebraic model for nonisotropic turbulent dissipation rate in Reynolds stress closures

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A method to account for effects of an anisotropic dissipation rate in second-order moment closures of turbulent flows is presented. The modeled transport equations for the Reynolds stresses and the scalar dissipation rate are supplemented by nonlinear algebraic relations, in which the anisotropies of the dissipation rate tensor are expressed in the anisotropies of the Reynolds stress tensor. Symmetry and other constraints reduce the number of undetermined coefficients to one. It is shown that the model correctly describes the initial behavior of a suddenly distorted turbulence provided that the model parameter assumes a value of $\frac{3}{4}$. The model is compared with full numerical simulations of three basic types of strained homogeneous turbulence, and is shown to give good agreement with these data. The present approach of isolating the effects of anisotropic dissipation rate in second-order moment closures is a step toward improved modeling of the energy redistribution terms in the Reynolds stress transport equations, and could also be used for improved modeling of the scalar dissipation rate equation.

I. INTRODUCTION

Turbulent boundary layers, channel flows, and other wall-bounded flows exhibit a strongly nonisotropic energy distribution in the near-wall region. This is also true far away from walls when the flow is, or has been, subjected to mean strain or rotation. The nonisotropic character of such flows gives rise to difficult problems in the realm of turbulence modeling. The problem of how to account for the mechanisms leading to anisotropy and those responsible for the relaxation from anisotropic states is at present a central issue in the development of second-order moment closure models of turbulence.

In the context of Reynolds stress transport equations,

$$\frac{D\,\overline{u_i u_j}}{Dt} = P_{ij} - \varepsilon_{ij} + \Pi_{ij} + D_{ij}, \qquad (1)$$

where P_{ij} denotes the production rate term, ε_{ij} the dissipation rate term, Π_{ij} the pressure-strain-rate term, and D_{ij} the transport (diffusion) terms, only the production rate term is explicit in the Reynolds stresses whereas the other terms need to be modeled. In order to experimentally study the intercomponent energy transfer due to the pressure-strain-rate terms, and the related effects of an anisotropic dissipation rate, it is essential to choose a flow situation with a high degree of anisotropy albeit without many of the complexities of, for example, wall-bounded turbulent shear flows. In homogeneous flows such as flow through a contraction or an expansion, or turbulence relaxing from an initially anisotropic state (e.g., downstream of a contraction or an expansion), near-wall effects are absent and diffusion terms are negligible. Hence they are well suited for this type of experiments. Today, a feasible alternative to experiments, at least at low turbulent Reynolds numbers, is the use of comparisons with results of direct numerical flow simulations (see, e.g., Lee and Reynolds[1]). Some recent detailed studies of the pressure-strain-rate terms, and the underlying intermittent flow structures contributing to the spatially averaged mean value, have been made through the use of computer-generated databases for a turbulent shear flow (Brasseur and Lee[2]). For instance, it is shown that the correlation between the "rapid" and "slow" parts of the fluctuating pressure is very low due to a large difference in scales. This supports the idea of decomposition of the pressure-strain-rate term into two parts, as is commonly used in turbulence models.

Recent hot-wire measurements of grid-generated turbulence that was subjected to the mean strain field of axisymmetric contractions and subsequently allowed to relax toward isotropy in a straight duct (Hallbäck, Groth, and Johansson[3]), were focused on the determination of the dissipation rate of the streamwise velocity component. Hereby, the corresponding pressure-strain-rate term could be determined from the Reynolds stress transport equations. If the dissipation rate components are not measured separately (which can be experimentally complicated) the pressure-strain-rate terms can still be estimated by the use of some assumption about the distribution of the scalar dissipation rate ($\varepsilon \equiv \frac{1}{2}\varepsilon_{ii}$) among the individual components. When the approximation of an isotropic dissipation rate is used, nonisotropic dissipation rate effects are lumped together with those of the intercomponent energy transfer, which can lead to erroneous conclusions about modeling constants and the validity of different types of modeling approaches. This is evidenced in the literature from experimental determination of the Rotta constant[4] in the "slow," or return, part (Π^1_{ij}) of the pressure-strain-rate term (see, e.g., Launder, Reece, and Rodi[5] or Morris[6]). The experimental data of the strongly strained flows in Ref. 3 (Reynolds number $\mathrm{Re}_\Lambda \equiv 4k^2/\nu\varepsilon \approx 1000$, where k is the turbulent kinetic energy) show that the small scale structure of the turbulence becomes nonisotropic in the contractions, and that this anisotropy persists far downstream in the relaxation section. It

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was also shown that the anisotropy of the dissipation rate tensor was comparable in magnitude to that of the Reynolds stress tensor. This, in turn, shows that in experimental evaluations of turbulence models it is important to distinguish between the effects of anisotropic dissipation and intercomponent energy transfer.

In most turbulence models used today, the dissipation rate tensor is taken to be isotropic, and only a single transport equation for the scalar dissipation rate is treated in Reynolds stress closures and eddy visocity models. This simple model has sometimes been replaced by a linear relation (see, e.g., Hanjalić and Launder[7]) between the dissipation rate and the Reynolds stress anisotropies (a_{ij}):

$$\varepsilon_{ij} = \varepsilon(\tfrac{2}{3}\delta_{ij} + E a_{ij}),$$

where

$$a_{ij} \equiv (\overline{u_i u_j}/k) - \tfrac{2}{3}\delta_{ij}. \tag{2}$$

The coefficient E is taken to be a function of, for instance, the turbulent Reynolds number Re_Λ and is supposed to tend to zero (giving an isotropic dissipation rate) when Re_Λ tends to infinity and to unity when Re_Λ tends to zero.

Another approach is to include the effects of anisotropic dissipation rate in the return pressure-strain-rate term, giving a "return-to-isotropy" tensor T_{ij}. A model based on this concept was proposed by Lumley and Newman,[8] and was further investigated by Shih, Mansour, and Chen.[9] In the latter study, comparisons with data from direct numerical simulations of homogeneous shear flows and irrotationally strained flows were made, which indicated improved performance over earlier models.

One motivation for constructing a composite model of the return part of the pressure-strain rate and of the anisotropic part of the dissipation rate is the difficulty in determining the individual components of the dissipation rate tensor. However, as this is now experimentally feasible in some homogeneous turbulent flows, and certainly for low Reynolds number flows through full numerical simulation of the Navier–Stokes equations, separate modeling of the pressure-strain-rate and the dissipation rate terms ought to be a preferable approach since these two terms represent physically different phenomena (e.g., as a wall is approached Π_{ij} and ϵ_{ij} behave in quite different ways). Hereby, there is a greater hope for improved generality of Reynolds stress closures. In the present study an algebraic model for the dissipation rate tensor is proposed, which couples the anisotropies of ε_{ij} to those of the Reynolds stress tensor. Some comparisons between the proposed relation and results from direct numerical simulations are also presented.

II. AN ALGEBRAIC NONISOTROPIC DISSIPATION RATE MODEL

The approach adopted here is that of treating the effects of anisotropic dissipation rate through algebraic relations and it bears some resemblance to the approach behind algebraic Reynolds stress models. In the latter case the transport equations for k and ε are supplemented by algebraic relations for the individual components of the stress tensor, or equivalently, a_{ij}. In the present approach the modeled transport equations for the Reynolds stresses (or equivalently, those for k and a_{ij}) and the scalar dissipation rate ε are complemented by algebraic relations for the dissipation rate anisotropies e_{ij},

$$e_{ij} \equiv (\varepsilon_{ij}/\varepsilon) - \tfrac{2}{3}\delta_{ij}, \tag{3}$$

where

$$\varepsilon_{ij} = 2\nu\ \overline{u_{i,k} u_{j,k}}.$$

The latter relation is an exact definition of the dissipation rate components in homogeneous turbulence, but is approximately true in other flows as well when Re_Λ becomes large.

From the transport equations for the Reynolds stress anisotropies,

$$\frac{Da_{ij}}{Dt} = \mathscr{P}_{ij}(a_{kl}, U_{m,n}) + \frac{\varepsilon}{k}(a_{ij} - e_{ij}) + \frac{\Pi_{ij}}{k}, \tag{4}$$

where $\mathscr{P}_{ij}$ is the production of a_{ij}, it is seen that if the history of the anisotropies a_{ij} were known $\Pi_{ij} - \varepsilon e_{ij}$ could be determined. Hence e_{ij} (and Π_{ij}) can be written as functionals of k, a_{ij}, ε, the mean velocity derivatives $U_{i,j}$ and ν (included for generality, although not explicitly present in the equations). The definition of ε_{ij} makes it reasonable to assume that mean velocity gradients enter the functional relationship for e_{ij} only as invariants of $U_{i,j}$. Expanding the functionals in time and assuming that changes in the system are sufficiently slow relative to the memory time of the turbulence, these functionals can be reduced to functions of the present state (Lumley and Newman[8]). Thus the present aim is to find a model expressing e_{ij} as a tensorially correct function of a_{kl}.

From the definition of e_{ij} it is clear that the model expression must be symmetric in the indices i and j and have a zero trace. The most general expression, given by invariant theory (Lumley[10]), is

$$e_{ij} = f(\mathrm{II}_a, \mathrm{III}_a, \mathrm{Re}_\Lambda, S_2^*, S_3^*, \Omega_2^*) a_{ij} + g(\mathrm{II}_a, \mathrm{III}_a, \mathrm{Re}_\Lambda, S_2^*, S_3^*, \Omega_2^*)(a_{ik}a_{kj} - \tfrac{1}{3}\mathrm{II}_a \delta_{ij}), \tag{5}$$

where

$$\mathrm{II}_a = a_{ik}a_{ki}, \quad \mathrm{III}_a = a_{ik}a_{kl}a_{li},$$
$$S_2^* = \sqrt{2S_{ij}S_{ji}}\,k/\varepsilon, \quad S_3^* = \sqrt[3]{S_{ij}S_{jk}S_{ki}}\,k/\varepsilon,$$
$$\Omega_2^* = \sqrt{-2\Omega_{ij}\Omega_{ji}}\,k/\varepsilon, \quad S_{ij} = (U_{i,j} + U_{j,i})/2$$
$$\Omega_{ij} = (U_{i,j} - U_{j,i})/2;$$

II_a and III_a are the only independent invariants of a_{ij}.

Here the aim is to develop an explicit model relating the anisotropies of the dissipation rate tensor to those of the Reynolds stresses. A natural method of proceeding from Eq. (5) would be to expand f and g in power series of the invariants of a_{ij}, keeping in mind that II_a and III_a are of second and third order in the a_{ij} amplitudes. A direct approach, which automatically yields the correct number of model parameters, is to construct a general series expansion of e_{ij} in "powers" of a_{kl}. Such a general series expansion that satisfies the symmetry condition $e_{ij} = e_{ji}$, and the zero trace condition $e_{ii} = 0$, can be written

$$e_{ij} = c_1 a_{ij} + c_2(a_{ik}a_{kj} - \tfrac{1}{3}\mathrm{II}_a\,\delta_{ij}) + c_3(a_{ik}a_{kl}a_{lj} - \tfrac{1}{3}\mathrm{III}_a\,\delta_{ij})$$

$$+ c_4(a_{ik}a_{kl}a_{lm}a_{mj} - \tfrac{1}{3}\mathrm{IV}_a\,\delta_{ij})$$
$$+ c_5(a_{ik}a_{kl}a_{lm}a_{mn}a_{nj} - \tfrac{1}{3}\mathrm{V}_a\,\delta_{ij}) + \cdots, \qquad (6)$$

where $\mathrm{IV}_a = a_{ik}a_{kl}a_{lm}a_{mi}$, $\mathrm{V}_a = a_{ik}a_{kl}a_{lm}a_{mn}a_{ni}$, etc., are higher-order invariants of a_{ij}. The coefficients c_i can be functions of Re_Λ, S_2^*, S_3^*, and Ω_2^* but not of the a_{ij} invariants since all dependence of a_{ij} and its invariants are explicitly accounted for in (6). A similar procedure has also been used to find a nonlinear model for the return part of the pressure-strain term.[11]

The so-called Cayley–Hamilton theorem [also used in the derivation of (5)] states that only the two first powers and the three first invariants of a tensor in three-dimensional space are linearly independent (Cayley[12]). Here, the first invariant, the trace, is zero, which makes it possible to express all terms of order three and higher in combinations of $a_{ij}, a_{ik}a_{kj}, \mathrm{II}_a$, and III_a. The use of the Cayley–Hamilton identity,

$$a_{ik}a_{kl}a_{lj} - \tfrac{1}{3}\mathrm{III}_a\,\delta_{ij} = \tfrac{1}{2}\mathrm{II}_a\,a_{ij}, \qquad (7)$$

which, if repeatedly multiplied with the tensor a_{ij}, yields relations for the fourth- and higher-order terms, gives

$$e_{ij} = \{c_1 + c_3\tfrac{1}{2}\,\mathrm{II}_a + c_4\tfrac{1}{3}\,\mathrm{III}_a + c_5\tfrac{1}{4}\,\mathrm{II}_a^2 + \cdots\}a_{ij}$$
$$+ \{c_2 + c_4\tfrac{1}{2}\,\mathrm{II}_a + c_5\tfrac{1}{3}\,\mathrm{III}_a + \cdots\}(a_{ik}a_{kj} - \tfrac{1}{3}\,\mathrm{II}_a\delta_{ij}). \qquad (8)$$

This expression is seen to be of the same type as (5) but here the interrelations between higher-order terms in f and g are explicitly accounted for. A relation of the type (5) or (8) may be regarded as an attempt of relating the small scale anisotropies (coupled to e_{ij}) to the anisotropies residing in the large scales (coupled to a_{ij}).

In the limit of homogeneous two-component turbulence (in which the third velocity component has negligible energy) it follows from the definition of ϵ_{ij} that

$$e_{\underline{ii}} = a_{\underline{ii}} \quad \text{if } a_{\underline{ii}} = -\tfrac{2}{3}, \qquad (9)$$

where underscore signifies suppression of the summation rule. This condition stems from the fact that if a velocity component lacks energy, in homogeneous turbulence, then all its spatial derivatives must also vanish. As an example of a situation where (9) might occur, one may mention the case of a rapidly and strongly axisymmetrically strained flow where the streamwise fluctuating velocity component tends to zero. In fact the relation (9) is valid also for the wall normal velocity component in the inhomogeneous turbulent flow in the immediate vicinity of a wall. From (9) it is also evident that if the relative energy content is zero for two of the components, the third must be responsible for all the energy and all of the dissipation rate.

Relation (9) can be seen as a form of "boundary" or limiting condition that must be fulfilled for any generally valid model for the anisotropy measures e_{ij}. The condition (9) can also be viewed as a realizability condition for the dissipation rate term, which ensures that the energies remain positive.

If one terminates the series expansion (8) at the zeroth-order term, by setting c_k to zero for $k \geqslant 1$, one obtains the usual isotropic model ($e_{ij} = 0$), whereas termination at the first-order term gives the linear relation $e_{ij} = c_1 a_{ij}$ [i.e., Eq. (2)]. The zeroth-order expansion cannot satisfy the limiting condition, and if the linear relation is to satisfy the condition (9), c_1 must equal unity. It is easily shown that truncation at the second-order term also returns the linear model. Hence, at least cubic terms are needed in order to satisfy the limiting condition (9) while still retaining adjustment possibilities of the remaining model parameters.

Truncation of the series expansion at the third-order term, i.e., setting c_k to zero for $k \geqslant 4$, and application of the conditions of the two-component turbulence limit, which eliminates two of the coefficients, yield the following expression with only one free parameter:

$$e_{ij} = [1 + \alpha(\tfrac{1}{2}\mathrm{II}_a - \tfrac{2}{3})]a_{ij} - \alpha(a_{ik}a_{kj} - \tfrac{1}{3}\mathrm{II}_a\,\delta_{ij}). \qquad (10)$$

This is the proposed algebraic dissipation model that relates the anisotropies of the dissipation rate tensor to those of the Reynolds stress tensor. The value of α may be taken as a function of the turbulent Reynolds number (Re_Λ) and the strain-rate parameters S_2^*, S_3^*, and Ω_2^*. However, α may not depend on a_{ij} or its invariants.

III. DETERMINATION OF A NUMERIC VALUE OF THE MODEL PARAMETER α

The numeric value of α can be bounded by imposing various constraints on the behavior of the model. One basic constraint is that the degree of anisotropy of the dissipation rate should be less than or equal to that of the Reynolds stresses, which can be expressed as

$$\mathrm{II}_e \leqslant \mathrm{II}_a. \qquad (11)$$

From Eq. (10) it can be shown that this implies that α must lie in the range from zero to two. Another constraint is that of requiring $e_{\underline{ii}}$ to be a monotonous function of $a_{\underline{ii}}$. This can be shown to give a lower bound of zero and an upper bound of unity on α, hence

$$0 \leqslant \alpha \leqslant 1. \qquad (12)$$

One possibility of determining a specific value of α is to impose the requirement that the model should correctly predict the dissipation rate anisotropy for small times of initially isotropic turbulence subjected to a rapid strain. This is a physically relevant case, which can be regarded as a model, for instance, of turbulence entering a contraction or subjected to a plane strain.

Under such circumstances viscous and nonlinear effects are negligible for small enough times, which implies that the initial response of the turbulence can be described by rapid distortion theory (RDT) (see, e.g., Ref. 13). This method of determining the model coefficient is similar to that of Launder, Reece, and Rodi[5] who used the results of Crow[14] for a small sudden distortion to fix a parameter in a model for the rapid pressure-strain term.

The analysis for an arbitrary strain is given in the Appendix. The RDT analysis yields that, for small total rapid strains, the diagonal components of the dissipation rate anisotropy tensor are exactly half of the corresponding Reynolds stress anisotropies,

$$e_{ii} = \tfrac{1}{2}a_{ii}. \quad (13)$$

Note that this holds for a rapid irrotational strain as well as for a strong homogeneous shear. If in the latter case the coordinate system is chosen so that $U_1 = U(x_2)$, only a_{12} and e_{12} will be affected to first order in the (small) total shear and

$$e_{12} = \tfrac{1}{2}a_{12}. \quad (14)$$

Hence initially the dissipation rate tensor is less anisotropic than the Reynolds stress tensor. For small distortions and thereby small anisotropies the model Eq. (10) can be linearized:

$$e_{ij} = (1 - \tfrac{2}{3}\alpha)a_{ij}. \quad (15)$$

In order for this relation to be compatible with (13) and (14), and hence for the model equation to correctly describe the initial behavior of the dissipation rate anisotropies in the case of suddenly distorted isotropic turbulence, we find that

$$\alpha = \tfrac{3}{4}. \quad (16)$$

Since the expressions for the anisotropy measures are linear in the small total strains and total shears the results can be superimposed, and we may conclude that with $\alpha = \frac{3}{4}$ the model (10) will correctly predict the initial variation of the dissipation rate anisotropies for an arbitrary, not necessarily irrotational, rapid strain.

It may here be noted that for an axisymmetrically rapidly strained contracting flow one finds similar trends for the RDT results and the model prediction, not only for small total strains and anisotropies, but over the whole range from isotropy to the two-component limit (this is not the case for large rapid expansions though).

Figure 1 shows the relation between dissipation rate and Reynolds stress anisotropies predicted by the model (10) for axisymmetric turbulence. In this case there is only one independent anisotropy measure of the Reynolds stress tensor, here chosen as a_{11}, and we have $a_{22} = a_{33} = -\frac{1}{2}a_{11}$. Correspondingly, there is only one independent anisotropy measure (e.g., e_{11}) of the dissipation rate tensor. In order to give a picture of the sensitivity for the value of the model parameter, the prediction with $\alpha = 1$ is also included in Fig. 1. A lower value of α will give a curve closer to the straight line $e_{11} = a_{11}$, which is obtained by setting $\alpha = 0$.

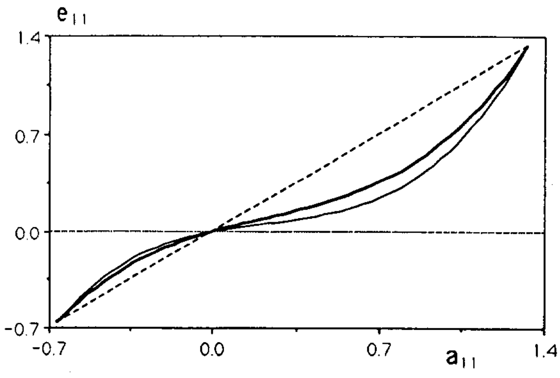

FIG. 1. Model (10) in an axisymmetric flow. The thick curve has $\alpha = \frac{3}{4}$, the thin curve $\alpha = 1$ and the dashed line $\alpha = 0$. Isotropic dissipation rate is given by $e_{11} \equiv 0$.

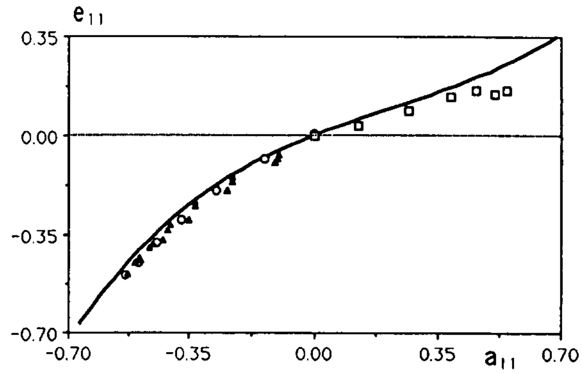

FIG. 2. e_{11} vs a_{11} for turbulence subjected to axisymmetric strain. Comparison between model (10) with $\alpha = \frac{3}{4}$ (thick curve) and numerically simulated results. Data from Ref. 1: contraction (○) and expansion (□) [$S_2^*(t=0) \approx 0.97$ resp. 0.71, $\mathrm{Re}_\Lambda \approx 50$]; Ref. 14: contraction (△) [S_2^* $(t=0) \approx 0.5$–2.0, $\mathrm{Re}_\Lambda \approx 15$–100]. Isotropic dissipation rate is given by $e_{11} \equiv 0$.

IV. COMPARISONS WITH FULL NUMERICAL SIMULATIONS

The proposed model for the anisotropy of the dissipation rate has been compared with results from three different full numerical simulations of homogeneous turbulence (Lee and Reynolds,[1] Rogallo,[15] Schumann and Patterson[16]). The first comparison is made for two axisymmetrically strained flows representing flow in a contraction[1,15] and in an expansion.[1] The mean strain-rate parameter S_2^* was approximately equal to unity in the cases in Ref. 1 and between 0.5 and 2 in Ref. 15. The Reynolds numbers Re_Λ were between 50 and 100. The model (10), with $\alpha = \frac{3}{4}$, is seen in Fig. 2 to give accurate predictions for the relations between e_{11} and a_{11} in both flows over a wide range of total strains and anisotropies.

Data for the relaxation phase "downstream" of axisymmetric strains are also available in Refs. 1 and 15, whereas in Ref. 16 the results are based on relaxations of undeveloped randomly generated axisymmetric turbulence fields. There is a fair agreement between the model and the various simulation cases (Fig. 3). The data from Refs. 15 and 16 all show similar trends rather independent of the initial conditions such as Reynolds number and previous strain rate.

The results in Figs. 2 and 3 substantiate the idea of approximating e_{ij} in terms of an expression in the anisotropies of the stress tensor. It is also apparent from the results that the proposed, rather simplistic, algebraic modeling approach has a potential for describing effects of anisotropic dissipation rate. However, in the axisymmetric cases there is only one independent anisotropy measure and an equally accurate prediction might be expected to result from a linear dissipation model in which E is allowed to vary in an appropriate manner with the invariants of a_{ij}. In a case with more than one independent anisotropy measure one cannot *a priori* expect a linear dissipation model, with the same value of the coefficient E for all tensor components, to be sufficient.

A more interesting test case may thus be that of homoge-

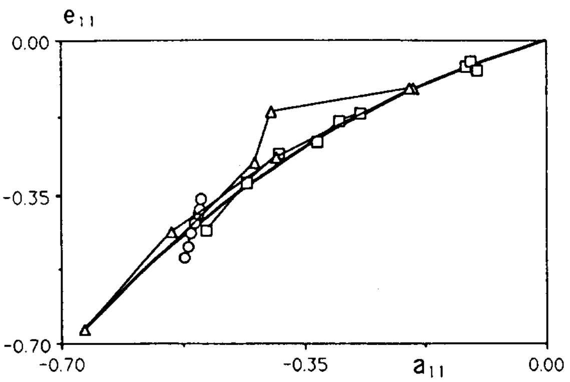

FIG. 3. e_{11} vs a_{11} for axisymmetric turbulence relaxing toward isotropy. Comparison between model (10) with $\alpha = \frac{3}{4}$ (thick curve) and numerically simulated results. Data from Ref. 1: (○) ($\mathrm{Re}_\Lambda \approx 45$); Ref. 14: (□) ($\mathrm{Re}_\Lambda \approx 50$–$70$); Ref. 15: (△) ($\mathrm{Re}_\Lambda \approx 85$–$840$). Symbols are linked together by straight lines for each relaxation case.

neous turbulence subjected to a plane strain. We here have two independent diagonal components of the a_{ij} tensor. Predictions for the dissipation rate anisotropies e_{11} and e_{33} are compared in Fig. 4 with results from the simulation data from Ref. 1 in which $a_{11} \approx 0$, i.e., $a_{33} \approx -a_{22}$. A rather good agreement is again seen to be achieved with a model coefficient value of $\frac{3}{4}$. Data from Ref. 15 with moderate strain rates ($S_2^* \approx 1$ resp. 2) agree well with the data in Fig. 4 and thus also with the model. The simulation data indicate that for very high strain rates ($S_2^* > 4$) the dissipation rate anisotropies depend more strongly on S_2^*. If α is taken as a constant (as done here), the accuracy of the model predictions would be reduced under such circumstances. Also included in Fig. 4 are predictions for the model (10) with $\alpha = 0$ (or equivalently $e_{ij} = a_{ij}$) and for the isotropic dissipation model (i.e., $e_{ij} = 0$). One should note that in this case no linear model

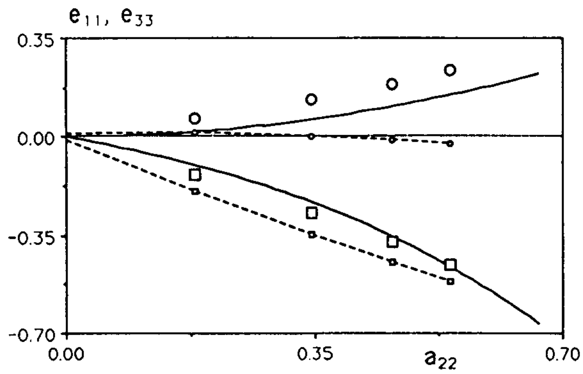

FIG. 4. e_{11} (○) and e_{33} (□) vs a_{22} for turbulence subjected to a plane strain. Comparison between model (10) with $\alpha = \frac{3}{4}$ (solid curves) and numerically simulated results from Ref. 1 [S_2^* ($t = 0$) ≈ 1, $\mathrm{Re}_\Lambda \approx 50$]. Dashed curves, with small symbols, represent the corresponding Reynolds stress anisotropy (or a model $e_{ij} = a_{ij}$). Isotropic dissipation rate is given by $e_{11} = e_{33} \equiv 0$.

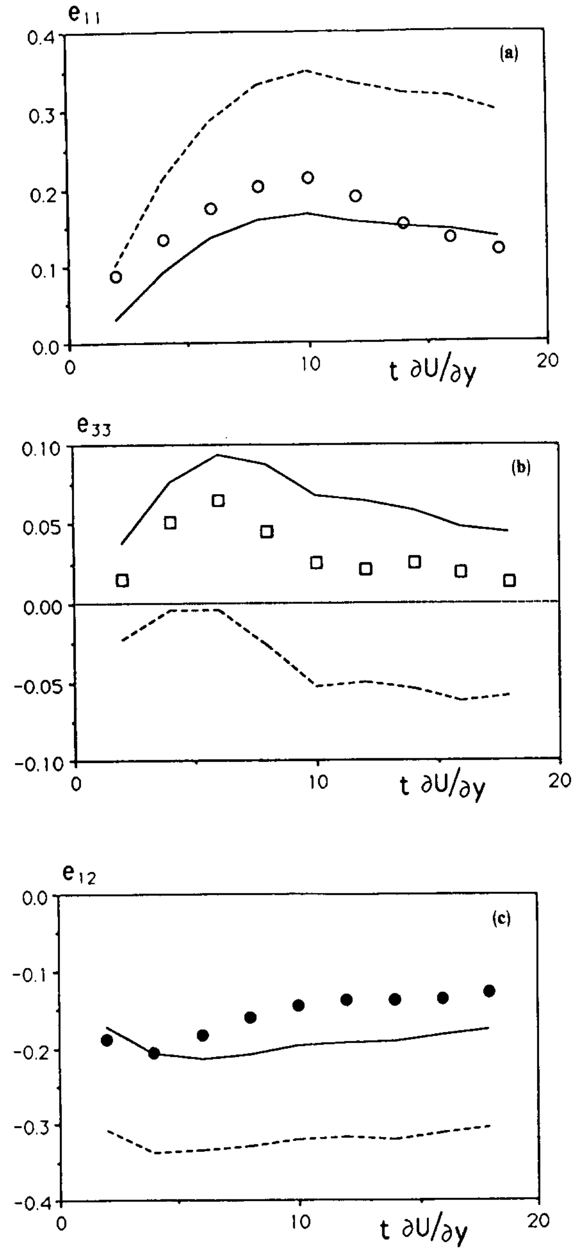

FIG. 5. Comparisons between model (10) predictions with $\alpha = \frac{3}{4}$ (solid curves) and numerically simulated results from Ref. 14 for homogeneous turbulent shear flow [$S_2^*(t=0) = \Omega_2^*(t=0) \approx 1.7$, $\mathrm{Re}_\Lambda \approx 300$–$2500$]. (a) e_{11}, (b) e_{33}, and (c) e_{12} versus total shear ($t\,\partial U/\partial y$). Dashed curves represent the corresponding Reynolds stress anisotropy (or a model $e_{ij} = a_{ij}$).

$e_{ij} = Ea_{ij}$, regardless of how E is prescribed to vary with the invariants, etc., is capable of predicting the dissipation rate anisotropies. This is obvious from the fact that a_{11} is nearly equal to zero while e_{11} increases with a_{22}. However, in the proposed model all components of the tensor a_{kl} enter in the expressions for the components of e_{ij} through the nonlinear term $\alpha(\frac{1}{2}\mathrm{II}_a\, a_{ij} - a_{ik}a_{kj} + \frac{1}{3}\mathrm{II}_a\, \delta_{ij})$. Hereby each e_{ij} is affected by the complete anisotropic state of the flow.

Another basic flow situation is that of turbulence subjected to homogeneous shear. This can be seen as a generic case for turbulent flows such as boundary layer and channel flow. In all these cases, which have a two-dimensional mean flow, there are three independent components of e_{ij}. In Fig. 5 e_{11}, e_{33}, and e_{12} are shown as functions of the total strain.[15] The 3,3- component is largely determined by the nonlinear term since a_{33} is nearly zero. For the other components the linear part ($e_{ij}^{\text{linear}} = 0.5a_{ij}$) dominates since the Reynolds stress anisotropies are of moderate amplitude.

The above comparisons with data from full numerical simulations of three homogeneous flow situations (axisymmetric flow, plane strain, and shear) indicate that the model, with a parameter value obtained from small rapid strain considerations, gives relatively good predictions even in cases with low and moderate strain rates. For large total strains at high strain rates the prediction capability of the model appears to be weakened. However, a correct behavior of the model for a component of vanishing energy (e.g., e_{33} in plane strain or e_{22} in shear) is always ensured by the limiting condition (9) for two-component turbulence.

V. COMPARISON WITH COMPOSITE "RETURN-TO-ISOTROPY" MODELS

The present model represents an attempt to explicitly account for effects of anisotropic dissipation rates. The success of this approach depends to some extent on the feasibility of obtaining reliable data for the dissipation rate components. Such data, although increasing in amount, are still rather scarce. In the Reynolds stress transport equations εe_{ij} plays a role similar to that of the so-called slow part of the pressure-strain-rate term (Π_{ij}^1). Composite models for the energy redistribution terms lump these two effects together. In the present notation the composite model used in Shih, Mansour, and Chen[9] (SMC) reads

$$\begin{aligned} T_{ij}^{\text{SMC}} &= \Pi_{ij}^1 - \varepsilon e_{ij} \\ &= -\varepsilon(\tfrac{1}{2}C_f F^{\xi} + 1)a_{ij} - \varepsilon(\gamma/4)\left[(\tfrac{2}{3} - \tfrac{1}{2}\text{II}_a)a_{ij} \right. \\ &\quad \left. + a_{ik}a_{kj} - \tfrac{1}{3}\text{II}_a\,\delta_{ij}\right], \end{aligned} \tag{17}$$

where $C_f f^{\xi}$ and γ are given functions of Reynolds number and the anisotropy tensor invariants. If this "return-to-isotropy" model is to be compared with the present e_{ij} -model, Eq. (10) must be complemented with a model for Π_{ij}^1. Because of the lack of detailed investigations of nonlinear explicit models for Π_{ij}^1, we choose here the commonly used linear Rotta[4] model and obtain

$$\begin{aligned} T_{ij}^{\text{JAM}} &= -c_1\varepsilon a_{ij} - \varepsilon e_{ij} \\ &= -\varepsilon(c_1 + 1)a_{ij} + \varepsilon\alpha\left[(\tfrac{2}{3} - \tfrac{1}{2}\text{II}_a)a_{ij} \right. \\ &\quad \left. + a_{ik}a_{kj} - \tfrac{1}{3}\text{II}_a\,\delta_{ij}\right], \end{aligned} \tag{18}$$

where c_1 is the Rotta constant. This should not be regarded as a proposed model for the energy redistribution tensor T_{ij}, but it allows interesting comparisons with the SMC model (despite the apparent inconsistency of using a linear model for Π_{ij}^1 together with a higher-order one for e_{ij}).

The two models (17) and (18) were tested against the experimental data of Ref. 17 on relaxation of strongly anisotropic homogeneous turbulence downstream of a contraction. The strain-rate parameter S_2^* in the experiment was about 3 and the turbulent Reynolds number, Re_Λ, about 2500. A rather good agreement was found between the SMC model prediction and the experimental data. A very close agreement was found for the present model if the Rotta constant was allowed to attain a value of $c_1 = 1.7$ (and $\alpha = \frac{3}{4}$). This value should be compared with $c_1 = 2.3$, representing the best choice of c_1 under the assumption of isotropic dissipation.

Direct comparison between (17) and (18) shows that c_1 and α correspond to $\frac{1}{2}C_f F^{\xi}$ and $-\gamma/4$, respectively. With the functions for $C_f F^{\xi}$ and γ in Ref. 9, $\frac{1}{2}C_f F^{\xi}$ will vary from about 2.1 to 0.85 and $-\gamma/4$ will be about 0.01 for the experimental case described above. The value of 0.01 seems to be very low as compared with the α used here, and it means that the higher-order terms in the second part of (17) have a negligible influence. The value of $-\gamma/4$ depends on the degree of anisotropy of a_{ij} and can never exceed 0.5 (the value reached in the two-component limit). If, in an axisymmetric case, $-\gamma/4$ is to be larger than, say, 0.1, then a_{11} must be smaller than -0.65, i.e., very close to the two-component limit, or greater than 1.25. Also, in all the irrotationally strained cases of Ref. 1 the value never exceeds 0.05. Thus Eq. (17) is essentially of the form $T_{ij} = Ca_{ij}$ except in highly anisotropic situations, e.g., in the immediate vicinity of a wall.

The above results should not in themselves be taken as an indication of the sufficiency of linear pressure-strain models. However, it possibly points to the advantage of separate modeling of anisotropic dissipation and pressure-strain rate for the construction of Reynolds stress models with as large a generality as possible.

VI. SUMMARY

A nonlinear algebraic model for the dissipation rate tensor has been derived, and tested against numerical simulation data. The set of algebraic relations for the anisotropy of the dissipation rate tensor should be regarded as a supplement to the transport equations for the individual Reynolds stress components (or k and a_{ij}) and the scalar dissipation rate, in the context of Reynolds stress transport models. A major advantage with the present approach is that the effects of anisotropic dissipation rate can be isolated and the modeling of these tested separately against experiments, physical or numerical. Another advantage is that knowledge of the anisotropy of the dissipation rate tensor could be used for improved modeling of the production term in the scalar dissipation rate transport equation.

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APPENDIX: RDT ANALYSIS OF THE RESPONSE OF INITIALLY ISOTROPIC TURBULENCE SUBJECTED TO A SUDDEN SMALL RAPID ARBITRARY DISTORTION

Rapid distortion theory can be used to predict the behavior of turbulence subjected to a suddenly applied rapid irrotational strain rate. Here we derive an expression for the relation between the Reynolds stress and dissipation rate anisotropies under such circumstances. The quantities needed are thus $\overline{u_i u_j}$ and $\overline{u_{i,n} u_{j,n}}$ as functions of the strain tensor. The presentation closely follows that of Ref. 13, pp. 68–73.

We begin by introducing an energy spectrum tensor $\Phi_{ij}(\mathbf{k})$ ($\mathbf{k}$ is the wave-number vector) for the homogeneous turbulence defined by

$$\overline{u_i(\mathbf{x}+\mathbf{r})u_j(\mathbf{x})} = \iiint \Phi_{ij}(\mathbf{k}) e^{i\mathbf{kr}}\, d\mathbf{k}. \quad \text{(A1)}$$

From this definitioin it is readily shown that (a prime is used to denote the initial state of the turbulence),

$$\overline{u_i u_j}' = \iiint \Phi'_{ij}(\mathbf{k}) d\mathbf{k}, \qquad \overline{u_{i,n} u_{j,n}}' = \iiint k^2 \Phi'_{ij}(\mathbf{k}) d\mathbf{k}. \quad \text{(A2)}$$

Hence the dissipation rate components can be written as

$$\varepsilon'_{ij} = 2\nu \iiint k^2 \Phi'_{ij}(\mathbf{k}) d\mathbf{k}. \quad \text{(A3)}$$

At a later instant when the flow has been distorted we have

$$\overline{u_i u_j} = \iiint \Phi_{ij}(\boldsymbol{\eta}) d\boldsymbol{\eta}, \qquad \varepsilon_{ij} = 2\nu \iiint \eta^2 \Phi_{ij}(\boldsymbol{\eta}) d\boldsymbol{\eta}, \quad \text{(A4)}$$

where in the rapid distortion limit it is possible to express $\boldsymbol{\eta}$ and Φ_{ij} as functions of $\mathbf{k}$, $\Phi'_{ij}(\mathbf{k})$, and the strain tensor. If a small arbitrary irrotational strain is imposed on the fluid the strain tensor will be diagonal with the components equal to $1+\delta_i\,(i=1,2,3)$. Here δ_i can be interpreted as the relative extension of a fluid element in the ith direction, i.e., $\delta_i = \Delta l_i / l_i$, where l_i is the length of the undistorted fluid element. The small distortion is taken to be the result of a large strain rate acting during a short time, which motivates the applicability of RDT. In initially isotropic turbulence $\Phi'_{ij}(\mathbf{k}) = E(k)\,(k^2\delta_{ij} - k_i k_j)/(4\pi k^4)$ leading to the following expressions, valid to first order in the δ_i's, for $\boldsymbol{\eta}$ and Φ_{ij} (no implicit summation over repeated indices):

$$\boldsymbol{\eta} = \left(\frac{k_1}{1+\delta_1}, \frac{k_2}{1+\delta_2}, \frac{k_3}{1+\delta_3}\right), \quad \text{(A5)}$$

$$\eta^2 \approx k^2 - 2\sum_{j=1}^{3} k_j^2 \delta_j, \quad \text{(A6)}$$

$$\Phi_{ii} \approx \frac{E(k)}{4\pi k^6}\left(k^2(k^2-k_i^2) - 2(k^4 - 3k^2k_i^2)\delta_i - 4k_i^2 \sum_{j=1}^{3} k_j^2 \delta_j\right), \quad \text{(A7)}$$

$$\sum_{i=1}^{3} \Phi_{ii} \approx \frac{E(k)}{2\pi k^2}\left(1 - \sum_{i=1}^{3}\delta_i + \frac{1}{k^2}\sum_{i=1}^{3} k_i^2 \delta_i\right), \quad \text{(A8)}$$

$$\eta^2 \Phi_{ii} \approx \frac{E(k)}{4\pi k^4}\left(k^2(k^2-k_i^2) - 2(k^4 - 3k^2k_i^2)\delta_i - 2(k^2+k_i^2)\sum_{j=1}^{3} k_j^2\delta_j\right), \quad \text{(A9)}$$

$$\sum_{i=1}^{3} \eta^2 \Phi_{ii} \approx \frac{E(k)}{2\pi k^2}\left(k^2 - k^2\sum_{i=1}^{3}\delta_i - \sum_{i=1}^{3} k_i^2 \delta_i\right). \quad \text{(A10)}$$

The continuity condition implies that $\Sigma_{j=1}^{3}\delta_j = 0$, i.e., $d\eta_1\, d\eta_2\, d\eta_3$ can be replaced by $dk_1\, dk_2\, dk_3$.

The expressions (A7)–(A10) are introduced into the integrals (A2) and (A3), which then are solved by use of spherical coordinates: $k_1 = k\cos\theta$, $k_2 = k\sin\theta\cos\varphi$, $k_3 = k\sin\theta\sin\varphi$. After some algebra we obtain the following for the initial response of isotropic turbulence subjected to a sudden rapid distortion (no summation over repeated indices):

$$\overline{u_i u_i} = \left(\frac{2}{3} - \frac{8}{15}\delta_i\right)\int_0^\infty E(k)dk,$$

$$\varepsilon_{ii} = \left(\frac{2}{3} - \frac{4}{15}\delta_i\right)2\nu\int_0^\infty k^2E(k)dk,$$

or

$$a_{ii} = -\tfrac{8}{15}\delta_i, \quad e_{ii} = -\tfrac{4}{15}\delta_i,$$

i.e., for small total strains we have the following relation between the dissipation rate and Reynolds stress anisotropies:

$$e_{ii} = \tfrac{1}{2}a_{ii}.$$

For small distortions and thereby small anisotropies the algebraic dissipation rate anisotropy model [Eq. (10)], if linearized, reads

$$e_{ij} = (1 - \tfrac{2}{3}\alpha)a_{ij}.$$

Hence the proposed model correctly describes the relation between large and small scale anisotropies (between a_{ij} and e_{ij}) for the case of small rapid distortions provided that

$$\alpha = \tfrac{3}{4}.$$

A similar analysis can also be carried out for a homogeneous shear flow using the theory of weak turbulence (see, e.g., Ref. 18, pp. 71–85). If the coordinate system is chosen so that $U_1 = U(x_2)$, and a_{12} and e_{12} will be affected to first order in the total shear $\sigma \equiv t\,\partial U/\partial x_2$:

$$a_{12} = -\tfrac{4}{15}\sigma, \quad e_{12} = -\tfrac{2}{15}\sigma.$$

We see that this is consistent with the above value of $\frac{3}{4}$ for the model parameter α.

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